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Duality in Multivalued Complementarity Theory by Using Inversions and Scalar Derivatives

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Abstract. We present in this paper several results related to the study of multivalued complementarity problems. Our results are based on the notions of exceptional family of elements and infinitesimal exceptional family of elements. A duality between these notions and the scalar derivatives are also used. The duality is achieved by using inversions.

Key words: Exceptional family of elements, Infinitesimal exceptional family of elements, Inversions, Multivalued complementarity problems, Scalar derivatives

1. Introduction

The complementarity theory is now in developing. The main goal of this relatively new domain of Applied Mathematics is the study of complementarity problems. In many practical problems the complementarity problems are related to the study of equilibrium; equilibrium as it is considered in Physics, in Technics and also the equilibrium of economical systems.

There exists four classes of complementarity problems: (1) explicit complementarity problems; (2) implicit complementarity problems; (3) complementarity problems with respect to an ordering and (4) multivalued complementarity problems. The *multivalued complementarity problems* are considered because in many practical problems instead of single-valued mappings set-valued mappings arise. The set-valued mappings are related to the presence of perturbations in the approximate definition of function values or to the uncertainty in mathematical models. While many results have been obtained for complementarity problems defined by single-valued mappings, there are relatively few papers dedicated to complementarity problems defined by set-valued mappings (see [1–14]).

In this paper we present several results on multivalued complementarity problems by using the notions of *exceptional family of elements*, *infinitesimal exceptional family of elements* and *scalar derivatives*. Because the notion of infinitesimal exceptional family of elements, we can use the scalar derivative at the origin.

By a special inversion we introduce a *duality* between the *exceptional family of elements* and the *infinitesimal exceptional family of elements*. By using this duality we show how new classes of set-valued mappings for which the multivalued complementarity problem has a solution can be obtained. We used a similar duality in our papers (G. Isac and S.Z. Németh, submitted for publication) dedicated to the complementarity problems defined by single-valued mappings.

This paper emphasizes the effectiveness of the topological method based on the notion of exceptional family of elements introduced by the first author of this paper in [15] and applied in the study of complementarity problems and variational inequalities in [10, 15–29]. We note that the notion of exceptional family of elements is based on the Leray–Schauder type alternatives.

This work can be considered as a starting point of a new research direction in the study of multivalued complementarity problems.

2. Preliminaries

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. $K \subset H$ is called a closed pointed convex cone if the following conditions are satisfied:

1. $K + K \subseteq K$,
2. $\lambda K \subseteq K$ for all $\lambda \in R_+$,
3. $K \cap (-K) = \{0\}$.

A closed pointed convex cone K induces an *ordering* on H defined by $x \leqslant y$ if and only if $y - x \in K$. The *dual cone* of K is the closed convex cone K^* defined by

$$K^* = \{y \in H | \langle x, y \rangle \geqslant 0 \text{ for all } x \in K\}.$$

Since K is closed and convex, the projection operator $P_K : H \to K$ onto K is well defined by the equation

$$\|x - P_K(x)\| = \min_{y \in K} \|x - y\|.$$

It is known that for every $x \in H$, $P_K(x)$ is uniquely defined by the relations:

(i) $\langle P_K(x) - x, y \rangle \geqslant 0$ for all $y \in K$,
(ii) $\langle P_K(x) - x, P_K(x) \rangle = 0$.

All topological vector spaces in this paper are assumed to be real Hausdorff spaces. Let E, F be topological vector spaces, $X \subset E$ and $Y \subset F$.

Denote by ∂X, $\overline{X}$ and $co(X)$ the boundary, the closure and the convex hull of X, respectively and by $\mathcal{P}(X)$ the family of all non-empty subsets of X.

Let $f: X \to Y$ be a set-valued mapping, i.e., $f: X \to \mathcal{P}(Y)$. The mapping f is called *upper semicontinuous* (u.s.c.) on X if the set $\{x \in X \mid f(x) \subset V\}$ is open in X whenever V is an open subset of Y. f is said to be *compact* if $f(X)$ is relatively compact in Y.

A subset D of E is called *contractible* if there is a continuous mapping $h: D \times [0,1] \to D$ with $h(x,0) = x$ and $h(x,1) = x_0$, for some $x_0 \in D$.

We note that if D is convex, it is contractible since for any $x_0 \in D$ we can consider $h(x,t) = tx_0 + (1-t)x$. Similarly, a starshaped set at x_0 is contractible to x_0. If $M \subset X$ is a non-empty subset, we say that a continuous mapping $r: X \to M$ is a *retraction* if and only if $r(x) = x$ for all $x \in M$. In this case we say that M is a *retract* of X. A set $D \subset X$ is called a *neighborhood retract* if and only if D is a retract of some of its neighborhoods.

A compact metric space M is called an *absolute neighborhood retract* (ANR) if it has the universal property that every homeomorphic image of M in a separable metric space is a neighborhood retract. Every compact convex set in an Euclidean space is an absolute neighborhood retract. It is well known that if $f: X \to Y$ is (u.s.c) and $f(x)$ is compact for every $x \in K$, then for every compact subset D of X the set

$$f(D) = \bigcup_{x \in D} f(x)$$

is also compact [30].

If Ω is a lattice with a minimal element denoted by 0, a function $\Phi : \mathcal{P}(E) \to \Omega$ is called a *measure of non-compactness* provided that the following conditions hold for any $X_1, X_2 \in \mathcal{P}(E)$:

1. $\Phi(\overline{co}(X_1) = \Phi(X_1)$,
2. $\Phi(X_1) = 0$ if and only if X_1 is precompact,
3. $\Phi(X_1 \cup X_2) = \max\{\Phi(X_1), \Phi(X_2)\}$.

We say that a mapping $f: X \to Y$ is Φ-condensing if for all $A \subset X$ with $\Phi(f(A)) \geqslant \Phi(A)$, A is relatively compact.

A compact set-valued mapping $f: X \to E$ is Φ-condensing if either the domain X is complete or E is quasi-complete.

Every set-valued mapping defined on a compact set is Φ-condensing (see [31–33]).

3. Approachable and approximable mappings

Let $E(\tau)$ be a Hausdorff locally convex topological vector space, $\mathfrak{U}$ be a fundamental basis of convex symmetric neighborhoods of the origin and X, Y non-empty subsets of E. In this paper we suppose that $f: X \to Y$ is a set-valued mapping with non-empty values.

We say that a single-valued mapping $s: X \to Y$ is a *selection* of the set-valued mapping $f: X \to Y$ if for any $x \in X$, $s(x) \in f(x)$.

In [31, 34–38] were introduced and studied the following notions.

For given $U, V \in \mathfrak{U}$, a function $s: X \to Y$ is called a (U, V)-*approximable selection* of f if for any $x \in X$,

$$s(x) \in (f[(x + U) \cap X] + V) \cap Y.$$

The set-valued mapping $f: X \to Y$ is said to be *approachable* if it has a continuous (U, V)-approximable selection for any $(U, V) \in \mathfrak{U} \times \mathfrak{U}$.

Finally, we say that $f: X \to Y$ is *approximable* if its restriction $f\,|_D$ to any compact subset D of X is approachable. Examples of approachable and approximable mappings can be found in [31, 34–38]. Now we indicate a few examples.

If X is a topological space, Y a convex subset in a locally convex space and $f: X \to Y$ an u.s.c. with convex values, then f is approximable. If X is a compact ANR, Y is an ANR and the values of f are compact, then f is approachable.

4. Complementarity problem

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f: H \to H$ a set-valued mapping with non-empty-values.

We say that f is *completely upper semicontinuous* (c.u.s.c.) if it is upper semicontinuous and for any bounded set $B \subset H$ we have that $f(B) = \bigcup_{x \in B} f(x)$ is relatively compact.

We say that f is *projectionally* Φ-*condensing* (*projectionally approximable*) with respect to K if $P_K(f)$ is Φ-condensing (resp. approximable).

The *Multivalued Complementarity Problem* defined by f and the cone K is

$$\mathrm{MCP}(f, K): \begin{cases} \text{find } x_* \in K \text{ and} \\ x_*^f \in f(x_*) \cap K^* \text{ such that} \\ \langle x_*, x_*^f \rangle = 0. \end{cases}$$

5. Scalar derivatives

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $C \subseteq H$ a set which contains at least one non-isolated point; $F, G: C \to H$ be set-valued mappings and x_0 a non-isolated point of C. The following definition is an extension of Definition 2.2 [39]:

DEFINITION 5.1. The limit

$$\underline{F}^{\#}(x_0) = \liminf_{\substack{x \to x_0, x \in C \\ x^F \in F(x), x_0^F \in F(x_0)}} \frac{\langle x^F - x_0^F, x - x_0 \rangle}{\|x - x_0\|^2}$$

is called the *lower scalar derivative* of F at x_0. Taking lim sup in place of lim inf, we can define the upper scalar derivative $\overline{F}^{\#}(x_0)$ of F at x_0 similarly.

Definition 5.1 can be extended for the unordered pair of set-valued mappings (F, G). The idea was inspired by the notion of derivative of a function with respect to another function [40].

DEFINITION 5.2. The limit

$$\underline{(F,G)}^{\#}(x_0) = \liminf_{\substack{x \to x_0, x \in C \\ x^F \in F(x), x_0^F \in F(x_0) \\ x^G \in G(x), x_0^G \in G(x_0)}} \frac{\langle x^F - x_0^F, x^G - x_0^G \rangle}{\|x - x_0\|^2}$$

is called the *lower scalar derivative* of the unordered pair of set-valued mappings (F, G) at x_0. Taking lim sup in place of lim inf, we can define the upper scalar derivative $\overline{(F,G)}^{\#}(x_0)$ of (F, G) at x_0 similarly.

REMARK 5.1. If $G = I$, we obtain Definition 5.1.

Scalar derivatives were studied in [39, 41] and successfully applied to fixed point theorems in [42, 43] and to complementarity problems in [44, 45].

6. Inversions

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\| \cdot \|$ the norm generated by $\langle \cdot, \cdot \rangle$. The following definition is an extension of Example 5.1 p. 169 [46]:

DEFINITION 6.1. The operator

$$i : H \setminus \{0\} \to H \setminus \{0\}; \qquad i(x) = \frac{x}{\|x\|^2}$$

is called *inversion* (of pole 0).

It is easy to see that i is one to one and $i^{-1} = i$.

Let $K \subset H$ be a closed pointed convex cone and $f : K \to H$ a set-valued map. Since $K \setminus \{0\}$ is an invariant set of i the following definition makes sense.

DEFINITION 6.2. The *inversion* (of pole 0) of the set-valued mapping f is the set-valued mapping $\mathcal{I}(f) : K \to H$ defined by

$$\mathcal{I}(f)(x) = \begin{cases} \|x\|^2 (f \circ i)(x) & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0. \end{cases}$$

We can show that $\mathcal{I}(\mathcal{I}(f)) = f$.

The properties of inversions were studied in detail in [42].

7. Exceptional family of elements

The next definition can be found in [9].

DEFINITION 7.1. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ be a closed pointed convex cone and $f : H \to H$ a set-valued mapping. We say that a family of elements $\{x_r\}_{r>0} \subset K$ is an *exceptional family of elements* for f with respect to K, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ and an element $x_r^f \in f(x_r)$ such that the following conditions are satisfied:

1. $u_r = \mu_r x_r + x_r^f \in K^*$ for all $r > 0$,
2. $\langle u_r, x_r\rangle = 0$ for all $r > 0$,
3. $\|x_r\| \to +\infty$ as $r \to +\infty$.

The next theorem is Theorem 2 of [9].

THEOREM 7.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ an u.s.c set-valued mapping with non-empty values. If the following assumptions are satisfied*

1. *$x - f(x)$ is projectionally Φ-condensing, or $f(x) = x - T(x)$, where T is a c.u.s.c. set-valued mapping with non-empty values,*
2. *$x - f(x)$ is projectionally approximable and $P_K[x - f(x)]$ is with closed values,*

then there exists either a solution to the problem MCP(f, K), *or an exceptional family of elements for f with respect to K.*

The next definition can be found in [9].

DEFINITION 7.2. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone. We say that a set-valued mapping $f : H \to H$ with non-empty values satisfies *condition* Θ with respect to K if, there exists a real number $\rho > 0$ such that for each $x \in K$ with $\|x\| > \rho$ there exists $p \in K$ with $\|p\| < \|x\|$ such that $\langle x - p, x^f\rangle \geqslant 0$ for all $x^f \in f(x)$.

DEFINITION 7.3. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone. We say that a set-valued mapping $f : H \to H$ with non-empty values satisfies *condition* $\widetilde{\Theta}$ with respect to K if, there exists a real number $\rho > 0$ such that for each $x \in K$ there exists $p \in K$ with $\langle p, x\rangle < \|x\|^2$ such that $\langle x - p, x^f\rangle \geqslant 0$ for all $x^f \in f(x)$.

The next lemma shows that condition $\widetilde{\Theta}$ is an extension of condition Θ.

LEMMA 7.1. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $K \subset H$ *a closed convex cone and* $f: H \to H$ *a set-valued mapping with non-empty values. If* f *satisfies condition* Θ *with respect to* K*, then it satisfies the condition* $\widetilde{\Theta}$ *with respect to* K*.*

Proof. Since f satisfies condition Θ with respect to K, there exists $\rho > 0$ such that for each $x \in K$ with $\|x\| > \rho$, there exists $p \in K$ with $\|p\| < \|x\|$ such that $\langle x - p, x^f\rangle \geqslant 0$ for all $x^f \in f(x)$. By the Cauchy inequality

$$\langle p, x\rangle \leqslant \|p\|\|x\| < \|x\|^2.$$

Hence, f satisfies condition $\widetilde{\Theta}$ with respect to K. □

The next theorem is proved in [9].

THEOREM 7.2. *Let* H *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone and* $f: H \to H$ *a set-valued mapping with non-empty values. If* f *satisfies condition* Θ *with respect to* K*, then it is without exceptional family of elements with respect to* K*.*

THEOREM 7.3. *Let* H *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone and* $f: H \to H$ *a set-valued mapping with non-empty values. If* f *satisfies condition* $\widetilde{\Theta}$ *with respect to* K*, then it is without exceptional family of elements with respect to* K*.*

Proof. Suppose to the contrary, that f has an exceptional family of elements $\{x_r\}_{r>0} \subset K$ with respect to K. Since $\|x_r\| \to \infty$ as $r \to \infty$, we can choose a real number r such that $\|x_r\| > \rho$. By condition $\widetilde{\Theta}$ there exists $p_r \in K$ such that $\langle p_r, x_r\rangle < \|x_r\|^2$ and

$$\langle x_r - p_r, x^f\rangle \geqslant 0, \qquad \text{for all } x^f \in f(x_r) \tag{1}$$

By the definition of the exceptional family of elements there exists $\mu_r > 0$ and $x_r^f \in f(x_r)$ such that

$$\begin{cases} u_r = \mu_r x_r + x_r^f \in K^* \\ \text{and} \\ \langle u_r, x_r\rangle = 0 \end{cases} \tag{2}$$

By Equation (1) and (2) we have

$$\begin{aligned} 0 \leqslant \langle x_r - p_r, x_r^f\rangle &= \langle x_r - p_r, u_r - \mu_r x_r\rangle \\ &= \langle x_r - p_r, u_r\rangle - \mu_r\|x_r\|^2 + \mu_r\langle p_r, x_r\rangle \\ &\leqslant -\mu_r(\|x_r\|^2 - \langle p_r, x_r\rangle) < 0 \end{aligned}$$

which is a contradiction. □

8. Infinitesimal exceptional family of elements

DEFINITION 8.1. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space $K \subset H$ a closed pointed convex cone and $g: K \to H$ a set-valued mapping with non-empty values. We say that $\{y_r\}_{r>0} \subset K$ is an *infinitesimal exceptional family of elements* for g with respect to K, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ and an element $y_r^g \in g(y_r)$ such that the following conditions are satisfied:

1. $v_r = \mu_r y_r + y_r^g \in K^*$,
2. $\langle v_r, y_r \rangle = 0$,
3. $y_r \to 0$ as $r \to +\infty$.

DEFINITION 8.2. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone. We say that a set-valued mapping $g: H \to H$ with non-empty values satisfies *condition* ${}^i\Theta$ with respect to K if, there exists a real number $\lambda > 0$ such that for each $y \in K \setminus \{0\}$ with $\|y\| < \lambda$, there exists $q \in K$ with $\|q\| < \|y\|$ such that

$$\langle y - q, y^g \rangle \geqslant 0, \tag{3}$$

for all $y^g \in g(y)$.

DEFINITION 8.3. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone. We say that a set-valued mapping $g: H \to H$ with non-empty values satisfies *condition* ${}^i\widetilde{\Theta}$ with respect to K if, there exists a real number $\lambda > 0$ such that for each $y \in K \setminus \{0\}$ with $\|y\| < \lambda$, there exists $q \in K$ with $\langle q, y \rangle < \|y\|^2$ such that

$$\langle y - q, y^g \rangle \geqslant 0, \tag{4}$$

for all $y^g \in g(y)$.

The next lemma shows that condition ${}^i\widetilde{\Theta}$ is an extension of condition ${}^i\Theta$ and it can be proved similarly to Lemma 7.1.

LEMMA 8.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $g: H \to H$ a set-valued mapping with non-empty values. If g satisfies condition ${}^i\Theta$ with respect to K, then it satisfies condition ${}^i\widetilde{\Theta}$ with respect to K.*

THEOREM 8.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $g: H \to H$ a set-valued mapping with non-empty values. If g satisfies condition ${}^i\widetilde{\Theta}$ with respect to K, then it is without infinitesimal exceptional family of elements with respect to K.*

Proof. Suppose to the contrary, that g has infinitesimal family of elements $\{y_r\}_{r>0} \subset K$ with respect to K. For any $r > 0$ such that $\|y_r\| < \rho$ there is an element $q_r \in K$ with $\langle q_r, y_r\rangle < \|y_r\|^2$ satisfying relation (3), i.e.,

$$\langle y_r - q_r, y_r^g)\rangle \geqslant 0.$$

for an arbitrary $y_r^g \in g(y_r)$. Since, according to Definition 8.1, $\langle v_r, y_r\rangle = 0$ and $v_r \in K^*$, we have

$$\begin{aligned}0 &\leqslant \langle y_r - q_r, y_r^g\rangle = \langle y_r - q_r, v_r - \mu_r y_r\rangle \\ &= -\mu_r\|y_r\|^2 - \langle q_r, v_r\rangle + \mu_r\langle q_r, y_r\rangle \\ &\leqslant -\mu_r(\|y_r\|^2 - \langle q_r, y_r\rangle) < 0,\end{aligned}$$

which is a contradiction. □

REMARK 8.1. At first sight Theorem 8.1 seems to be a direct consequence of Theorems 9.2 and 9.4, proved in the next section. However, note that there might be an infinitesimal family of elements of g which contains zero.

COROLLARY 8.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $g: H \to H$ a set-valued mapping with non-empty values. If g satisfies condition ${}^i\Theta$ with respect to K, then it is without infinitesimal exceptional family of elements with respect to K.*

Proof. By Lemma 8.1 g satisfies condition ${}^i\widetilde{\Theta}$ with respect to K. Hence, by Theorem 8.1 g is without infinitesimal exceptional family of elements with respect to K. □

REMARK 8.2. At first sight Corollary 8.1 seems to be a direct consequence of Theorem 9.2 and 9.5, proved in the next section. However, note that there might be an infinitesimal family of elements of g which contains zero.

9. A duality and main results

THEOREM 9.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space $K \subset H$ a closed convex cone and $f: K \to H$ a mapping. Then $(x_*, x_*^f) \notin \{0\} \times H$ is a solution of* MCP(f, K) *if and only if (y_*, y_*^g) is a solution of* MCP(g, K), *where $y_* = i(x_*)$ is the inversions of x_*,*

$$y_*^g = \frac{1}{\|x_*\|^2} x_*^f$$

and $g = \mathcal{I}(f)$ is the inversion of f.

Proof. First we have to prove that $y_*^g \in g(y_*)$. Dividing both sides of the relation $x_*^f \in f(x_*)$ by $\|x_*\|^2$ we obtain

$$y_*^g \in \frac{1}{\|x_*\|^2} f(x_*),$$

which implies $y_*^g \in \|y_*\|^2 f(i(y_*)) = \mathcal{I}(f)(y_*) = g(y_*)$. It is easy to see that

$$\langle y_*, y_*^g \rangle = \frac{1}{\|x_*\|^4} \langle x_*, x_*^f \rangle. \tag{5}$$

and

$$\langle y_*^g, z \rangle = \frac{1}{\|x_*\|^2} \langle x_*^f, z \rangle, \tag{6}$$

for every $z \in K$. By using (5),

$$\langle x_*, x_*^f \rangle = 0$$

if and only if

$$\langle y_*, y_*^g \rangle = 0.$$

By using (6), $x_*^f \in K^*$ if and only if $y_*^g \in K^*$. □

THEOREM 9.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space $K \subset H$ a closed pointed convex cone and $f : K \to H$ a set-valued mapping with non-empty values. $\{x_r\}_{r>0} \subset K \setminus \{0\}$ is an exceptional family of elements for f with respect to K if and only if $\{y_r\}_{r>0} \subset K \setminus \{0\}$ is an infinitesimal exceptional family of elements for g with respect to K, where $y_r = i(x_r)$ and $g = \mathcal{I}(f)$.*

Proof. Bearing in mind the notations of Definition 8.1, we have

$$v_r = \mu_r y_r + y_r^g,$$

for some $y_r^g \in g(y_r)$. Hence,

$$v_r = \|y_r\|^2 \left(\mu_r i(y_r) + \frac{y_r^g}{\|y_r\|^2} \right).$$

Since $i^{-1} = i$, we have

$$v_r = \frac{1}{\|x_r\|^2} \left(\mu_r x_r + \|x_r\|^2 y_r^g \right). \tag{7}$$

Let

$$x_r^f := \|x_r\|^2 y_r^g. \tag{8}$$

We have $x_r^f \in f(x_r)$. Indeed,

$$x_r^f \in \|x_r\|^2 g(y_r) = \|x_r\|^2 \mathcal{I}(f)(y_r) = \|x_r\|^2 \|y_r\|^2 f(i(y_r)) = f(x_r).$$

Now let

$$u_r = \mu_r x_r + x_r^f. \tag{9}$$

Equations (7–9) imply that

$$v_r = \frac{1}{\|x_r\|^2} u_r.$$

Therefore,

$$\langle v_r, y_r \rangle = \frac{1}{\|x_r\|^4} \langle u_r, x_r \rangle \tag{10}$$

and

$$\langle v_r, z \rangle = \frac{1}{\|x_r\|^2} \langle u_r, z \rangle, \tag{11}$$

for every $z \in K$. Since $\|x_r\| \cdot \|y_r\| = 1$, $\|x_r\| \to +\infty$ if and only if $y_r \to 0$. By using (10),

$$\langle u_r, x_r \rangle = 0$$

if and only if

$$\langle v_r, y_r \rangle = 0.$$

By using (11), $u_r \in K^*$ if and only if $v_r \in K^*$. □

THEOREM 9.3. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone and* $f : H \to H$ *an u.s.c set-valued mapping with non-empty values such that*

1. $x - f(x)$ *is projectionally* Φ*-condensing, or* $f(x) = x - T(x)$, *where* T *is a c.u.s.c. set-valued mapping with non-empty values,*
2. $x - f(x)$ *is projectionally approximable and* $P_K[x - f(x)]$ *is with closed values,*
3. $f(0) \cap K^* = \emptyset$.

If every infinitesimal exceptional family of elements for $g = \mathcal{I}(f)$ *with respect to* K *contains* 0, *then the multivalued complementarity problem* MCP(f, K) *has a non-zero solution.*

Proof. Since $f(0) \cap K^* = \emptyset$, if MCP$(f, K)$ has a solution, then this solution is non-zero. By Theorem 7.1, it is enough to prove that f is without exceptional family of elements with respect to K. Suppose to the contrary that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to K. Since $f(0) \cap K^* = \emptyset$, by the definition of an exceptional family of elements $\{x_r\}_{r>0} \subset K \setminus \{0\}$. Hence, by Theorem 9.2, $g = \mathcal{I}(f)$ has an infinitesimal exceptional family of elements with respect to K which does not contain 0, which is a contradiction. □

THEOREM 9.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone, $f : H \to H$ a set-valued mapping with non-empty values and $g = \mathcal{I}(f)$. Then, f satisfies condition $\widetilde{\Theta}$ with respect to K if and only if g satisfies the condition ${}^i\widetilde{\Theta}$ with respect to K.*

Proof. Since $g = \mathcal{I}(f)$ and $\mathcal{I}(\mathcal{I}(f)) = f$, it follows that

$$f = \mathcal{I}(g). \tag{12}$$

Suppose that g satisfies condition ${}^i\widetilde{\Theta}$ with respect to K and prove that f satisfies condition $\widetilde{\Theta}$ with respect to K. Consider the constant λ of condition ${}^i\widetilde{\Theta}$ and let

$$\rho = \frac{1}{\lambda}.$$

Let $x \in K$ with

$$\|x\| > \rho, \tag{13}$$

$y = i(x)$ and $x^f \in f(x)$. Let $y^g = x^f/\|x\|^2$. We have $y^g \in g(y)$. Indeed, by (12) we have

$$y^g \in \frac{f(x)}{\|x\|^2} = \frac{\mathcal{I}(g)(x)}{\|x\|^2} = \frac{\|x\|^2 g(i(x))}{\|x\|^2} = g(y).$$

Since

$$\|y\| = \frac{1}{\|x\|},$$

it follows that $\|y\| < \lambda$. Hence, by condition ${}^i\widetilde{\Theta}$, there exists $q \in K$ with $\langle q, y\rangle < \|y\|^2$ such that

$$\langle y - q, y^g\rangle \geqslant 0. \tag{14}$$

Let

$$p = \frac{q}{\|y\|^2}. \tag{15}$$

Since $\langle q, y\rangle < \|y\|^2$ and $i^{-1} = i$, relation (15) implies that

$$\langle p, x\rangle = \frac{\langle q, y\rangle}{\|y\|^4} < \frac{1}{\|y\|^2} = \|x\|^2. \tag{16}$$

By (14) we also have,

$$\langle x - p, x^f\rangle = \|x\|^2\langle x - p, y^g\rangle = \|x\|^4\langle y - q, y^g\rangle \geqslant 0 \tag{17}$$

By (13), (16) and (17) f satisfies condition $\widetilde{\Theta}$ with respect to K.

Now, suppose that f satisfies condition $\widetilde{\Theta}$ with respect to K and prove that g satisfies condition ${}^i\widetilde{\Theta}$ with respect to K. Consider the constant $\rho > 0$ of condition $\widetilde{\Theta}$ and let

$$\lambda = \frac{1}{\rho}.$$

Let $y \in K \setminus \{0\}$ with $\|y\| < \lambda$. We have to prove that there exists $q \in K$ with $\langle q, y \rangle < \|y\|^2$ such that $\langle y - q, y^g \rangle \geqslant 0$, for all $y^g \in g(y)$. Since $f = \mathcal{I}(g)$, we can proceed as above. □

The next theorem can be proved similarly to Theorem 9.4.

THEOREM 9.5. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone, $f: H \to H$ a set-valued mapping with non-empty values and $g = \mathcal{I}(f)$. Then, f satisfies condition Θ with respect to K if and only if g satisfies condition ${}^i\Theta$ with respect to K.*

THEOREM 9.6. *Let H be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f: H \to H$ an u.s.c set-valued mapping with non-empty values such that*

1. *$x - f(x)$ is projectionally Φ-condensing, or $f(x) = x - T(x)$, where T is a c.u.s.c. set-valued mapping with non-empty values,*
2. *$x - f(x)$ is projectionally approximable and $P_K[x - f(x)]$ is with closed values.*

If $g = \mathcal{I}(f)$ satisfies condition ${}^i\Theta$ with respect to K, then the multivalued complementarity problem MCP(f, K) *has a solution.*

Proof. By Theorem 9.5, f satisfies condition Θ with respect to K. Hence, Theorems 7.1 and 7.2 implies that the multivalued complementarity problem MCP(f, K) has a solution. □

THEOREM 9.7. *Let H be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f: H \to H$ an u.s.c set-valued mapping with non-empty values such that*

1. *$x - f(x)$ is projectionally Φ-condensing, or $f(x) = x - T(x)$, where T is a c.u.s.c. set-valued mapping with non-empty values,*
2. *$x - f(x)$ is projectionally approximable and $P_K[x - f(x)]$ is with closed values.*

If $g = \mathcal{I}(f)$ satisfies condition ${}^i\widetilde{\Theta}$ with respect to K, then the multivalued complementarity problem MCP(f, K) *has a solution.*

Proof. By Theorem 9.4, f satisfies condition $\widetilde{\Theta}$ with respect to K. Hence, Theorems 7.1 and 7.3 implies that the multivalued complementarity problem MCP(f, K) has a solution.

THEOREM 9.8. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f: H \to H$ an u.s.c set-valued mapping with non-empty values such that*

1. *$x - f(x)$ is projectionally Φ-condensing, or $f(x) = x - T(x)$, where T is a c.u.s.c. set-valued mapping with non-empty values,*
2. *$x - f(x)$ is projectionally approximable and $P_K[x - f(x)]$ is with closed values.*

If there is a $\delta > 0$ and a mapping $h : B(0, \delta) \cap K \to K$ with $h(0) = 0$ and

$$\begin{cases} \overline{h}^{\#}(0) < 1, \\ \underline{(I - h, \mathcal{I}(f))}^{\#}(0) > 0, \end{cases}$$

where $B(0, \delta) = \{z \in H : \|z\| < \delta\}$, then the multivalued complementarity problem MCP(f, K) *has a solution.*

Proof. Let $g = \mathcal{I}(f)$. Since $\overline{h}^{\#}(0) < 1$, there is a λ_1 with $0 < \lambda_1 < \delta$ such that for every $y \in K$ with $\|y\| < \lambda_1$ we have

$$\langle h(y), y \rangle < \|y\|^2. \tag{18}$$

Since

$$\underline{(I - h, g)}^{\#}(0) > 0,$$

there is a λ_2 with $0 < \lambda_2 < \delta$ such that for every $y \in K$ with $\|y\| < \lambda_2$ we have

$$\langle y - h(y), y^g \rangle > 0, \tag{19}$$

for all $y^g \in g(y)$. Let $\lambda = \min\{\lambda_1, \lambda_2\}$. Obviously,

$$\lambda > 0. \tag{20}$$

For

$$\|y\| < \lambda \tag{21}$$

let $q = h(y)$. Then, relations (18) and (19) imply

$$\langle q, y \rangle < \|y\|^2. \tag{22}$$

and

$$\langle y - q, y^g \rangle \geqslant 0, \tag{23}$$

respectively for all $y^g \in g(y)$. Hence, relations (20)–(23) imply that g satisfies condition ${}^i\widetilde{\Theta}$. Hence, Theorem 9.7 implies that the problem MCP(f, K) has a solution.

In the particular case $h = 0$ we have as follows:

COROLLARY 9.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ an u.s.c set-valued mapping with non-empty values such that*

1. *$x - f(x)$ is projectionally Φ-condensing, or $f(x) = x - T(x)$, where T is a c.u.s.c. set-valued mapping with non-empty values,*
2. *$x - f(x)$ is projectionally approximable and $P_K[x - f(x)]$ is with closed values.*

If $\underline{\mathcal{I}(f)}^{\#}(0) > 0$, then the multivalued complementarity problem MCP(f, K) *has a solution.*

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